* Non-examinable comment:
Does the hermitian conjugate of A always exist? Is it unique? Yes and yes. We won't give rigorous arguments (which depend on the space of functions and the classes of operators considered). But we can give informal plausibility arguments.

For example, suppose our space of wavefunctions has a countable orthonormal basis $\{\psi_i\}_{i=1}^{\infty}$.

Then we can write the action of a general linear operator

$$\boxed{A \quad \text{as} \quad A\psi_i = \sum a_{ij} \psi_j}$$

complex coefficients

and represent

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} & \cdots \\ \vdots & & \vdots & \\ a_{n1} & \cdots & a_{nn} & \cdots \\ \vdots & & \vdots & \end{pmatrix}$$

as a countably infinite square matrix.

Define $A^\dagger = \begin{pmatrix} a_{11}^* & \cdots & a_{21}^* & \cdots \\ \vdots & \ddots & \vdots & \\ a_{1n}^* & \cdots & a_{nn}^* & \cdots \\ \vdots & & \vdots & \end{pmatrix} = (A^T)^*$ as in the finite dimensional case.

Then $(\psi_i, A\psi_j) = (\psi_i, a_{jk}\psi_k) = a_{ji}$

$(A^\dagger\psi_i, \psi_j) = (a_{ki}^*\psi_k, \psi_j) = a_{ki}(\psi_k, \psi_j)$
$= a_{ji}$

i.e. $(\psi_i, A\psi_j) = (A^\dagger\psi_i, \psi_j)$

and we can now verify $\boxed{(\phi_1, A\phi_2) = (A^\dagger\phi_1, \phi_2)}$

for general $\phi_1 = \sum c_{1i}\psi_i$, $\phi_2 = \sum c_{2i}\psi_j$

by anti-linearity and linearity of $(\, , \,)$.

To see uniqueness of $A^\dagger$, suppose

$$(A_1 \psi_1, \psi_2) = (\psi_1, A\psi_2) = (A_2 \psi_1, \psi_2)$$

for all wavefunctions ψ_1, ψ_2 and some operators $A_1 \neq A_2$.

Then $((A_1 - A_2)\psi_1, \psi_2) = 0 \quad \forall \psi_1, \psi_2$.

Since $A_1 - A_2 \neq 0$, there must be some ψ_1 such that $(A_1 - A_2)\psi_1 \neq 0$.

Take $\psi_2 = (A_1 - A_2)\psi_1$.

Then $((A_1 - A_2)\psi_1, (A_1 - A_2)\psi_1) = (\psi_2, \psi_2) = 0$.

But $\psi_2 \neq 0$. Contradiction since $(\,,\,)$ is positive definite.

* End of non-examinable comment